General Formulation of Space Frame Element Stiffness Matrix with Parabolic Tapered Section using Flexibility Matrix and Gauss Quadrature Numerical Integration

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Abstract: To achieve an optimized design of beams and columns, sometimes a tapered, curved element, honeycomb or truss beam will be used in the structural model. In recent development of aseismic steel structure, a reduced flange beam is used to control the location of energy dissipation. This approach will require development of new frame element stiffness matrix that due to its complexity may not have explicit equation available. In this paper, a general approach to derive a new stiffness matrix of 3D frame element with any longitudinal shape, void pattern and shear deformation will be given using Flexibility approach. Stiffness matrix derived from flexibility matrix using Symbolic math and Gauss Quadrature Numerical Integration are also given. Sample problems also given to verify the resulted stiffness matrix. Using this procedure, a new stiffness matrix can be derived for any section with any shape of tapered element.

Keywords: Tapered Section, Honeycomb section, Reduced Flange beam, Frame element stiffness, Explicit stiffness formulation, Gauss Quadrature Numerical Integration

Introduction

A tapered space frame element is a 3D space frame element with non-prismatic section. The section depth or width or both depth and width may varies along its length linearly or parabolically. Flange width can changes in certain shape as in a reduced flange beam, or the frame can have holes at its web, as in cellular beam or castelated beam. For a prismatic frame, the standard stiffness matrix can be defined by explicit equations. Due to its complexities and difficulties to derive the stiffness matrix for tapered section, the standard stiffness matrix usually will be modified from standard stiffness matrix using certain modification factor. This approach is a very simplified one, and can only be applied for certain case of tapered section. In this paper, a general formulation of tapered space frame element will be derived. The formulation can be used to derive the required stiffness matrix straightforwardly, resulted in explicit equation for simpler cases, or must be computed using numerical integration for more complex cases.

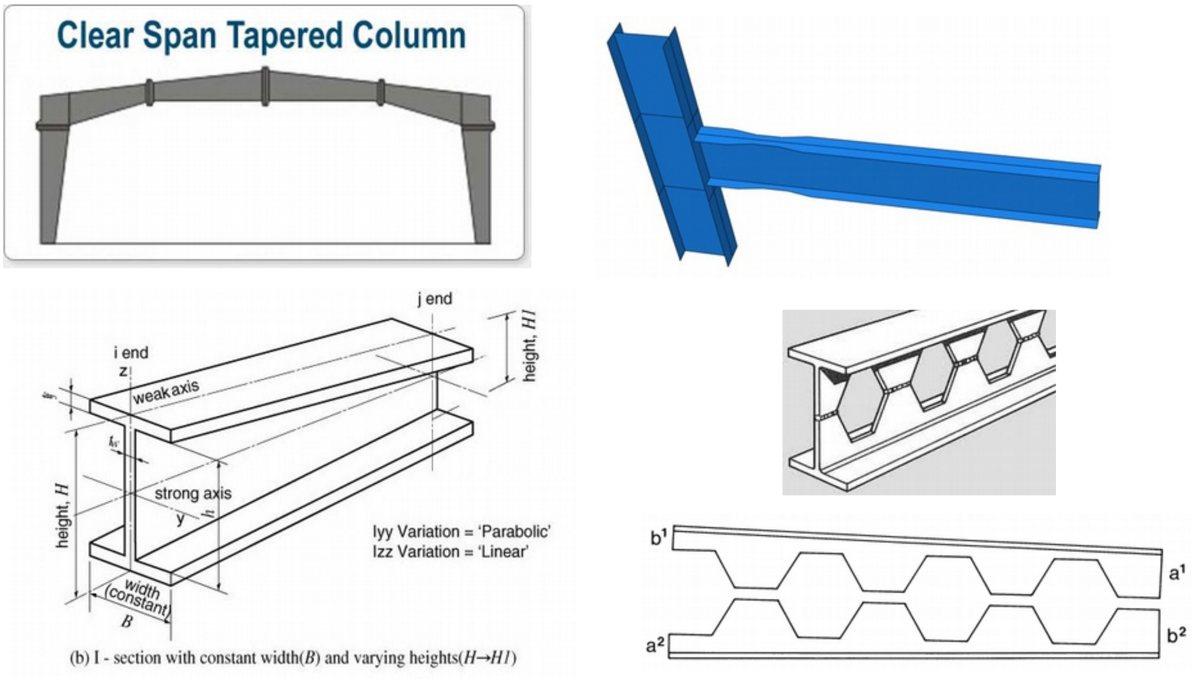


Fig.1 Several Tapered Frame Elements

Basics of The Flexibility Method (Force Method)

The Flexibility Method is a generalization of the Maxwell-Mohr method developed in 1864. In this approach, we will release the right node of a fixed end frame element, and apply a unit load in certain direction, and find the displacement in certain direction. The resulted displacement is the flexibility terms that can be used to build the flexibility matrix. The flexibility matrix then can be used to derived the required stiffness matrix using transformation matrix. Although the flexibility method is not suitable for computerized solution, it can be used to derived the flexibility terms easily even for complex sections, because the displacements can be computed using virtual energy approach, using the principle that external work must be equal to internal work. The internal work can be computed using integration of axial, bending, and shear stress and strain along the length of the member.

Member flexibility matrices will now be developed for non-prismatic members that are restrained at left node j and free at right node k. Direction of local member axis Ym is chosen so that major bending takes place in the Xm-Ym plane. Six kinds of end -actions are applied at right node k of the member : Unit load AM1,AM2,AM3 (positive in Xm,Ym,Zm directions), Unit torsion AM4 (positive in Xm direction), Unit bending moment AM5,AM6 (positive in Ym and Zm directions) shown in figure 2. For a space frame, the flexibility matrix will be a 6x6 matrix relating AM load vector to corresponding displacement Dm.

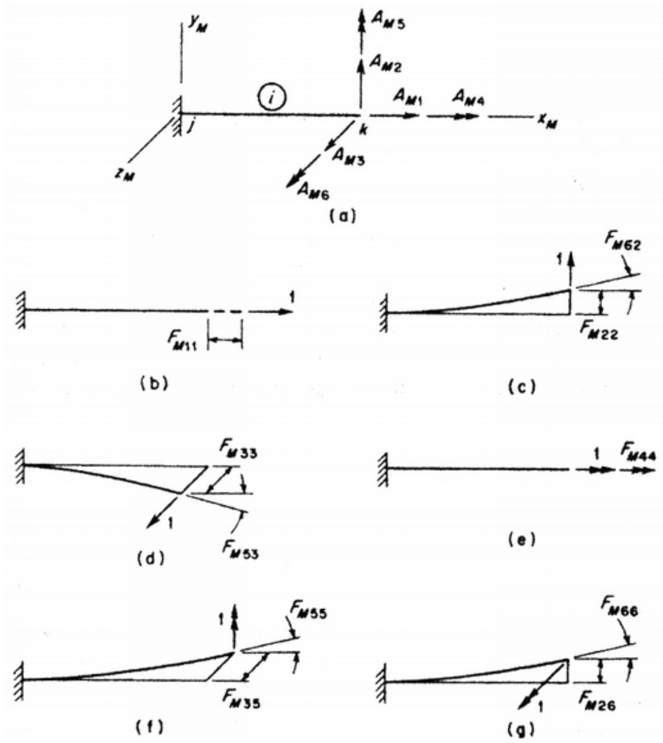


Fig.2. Flexibility terms for space frame element (Weaver, 1980)

For example, Fm11 is the axial displacement caused by a unit load in X direction, that can be computed as integration of axial stress and axial strain along the length of the element.

$$F_{M11} = \int_0^L \frac{1}{EA}\,dx \tag{1}$$

While Fm22 is a displacement in 2-2 direction (Ym direction) caused by a unit load acting in Ym direction that can be computed as integration of bending stress and bending strain along the length of the element. The second term is for the contribution of shear strain.

$$F_{M22} = \int_0^L \frac{(L-X)^2}{EI_z}\,dx + \int_0^L \frac{fy}{GA}\,dx \tag{2}$$

Using the same approach, the complete Flexibility terms will be :

1. $F_{M11} = \int_0^L \frac{1}{EA}\, dx$

2. $F_{M22} = \int_0^L \frac{(L-X)^2}{EI_z}\, dx + \int_0^L \frac{fy}{GA}\, dx$

3. $F_{M62} = \int_0^L \frac{(L-X)}{EI_z}\, dx = F_{M26}$

4. $F_{M33} = \int_0^L \frac{(L-X)^2}{EI_y}\, dx + \int_0^L \frac{fz}{GA}\, dx$

5. $F_{M53} = \int_0^L \frac{-(L-X)}{EI_y}\, dx = F_{M35}$ (3)

6. $F_{M44} = \int_0^L \frac{1}{GJ}\, dx$

7. $F_{M55} = \int_0^L \frac{1}{EI_y}\, dx$

8. $F_{M66} = \int_0^L \frac{1}{EI_z}\, dx$

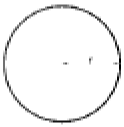
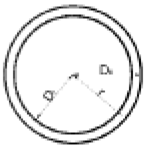
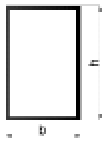
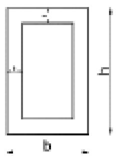
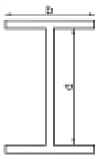
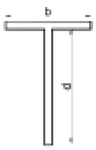
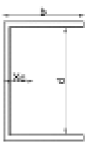
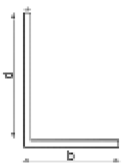
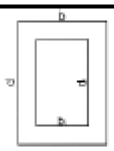
For standard prismatic element, values of A, Ix, Iy, Iz, fy, fz will be constants along the element, and the above integrals will give the familiar explicit equation of flexibility terms as follows (Weaver, 1980):

$$\mathbf{F}_{Mi} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3EI_z} & 0 & 0 & 0 & \frac{L^2}{2EI_z} \\ 0 & 0 & \frac{L^3}{3EI_Y} & 0 & \frac{-L^2}{2EI_Y} & 0 \\ 0 & 0 & 0 & \frac{L}{GJ} & 0 & 0 \\ 0 & 0 & \frac{-L^2}{2EI_Y} & 0 & \frac{L}{EI_Y} & 0 \\ 0 & \frac{L^2}{2EI_z} & 0 & 0 & 0 & \frac{L}{EI_z} \end{bmatrix}$$

Fig.3. The Familiar flexibility terms for a prismatic space frame element

For tapered frame elements, values of A, J, Iy, Iz, fy, fz will vary along the length of the element, so the values will contains terms of x, resulted in higher polynomial equation. Thus the explicit equation will be much complex and may not be available, in this case, numerical integral will be necessary to compute the terms.
Standard equation for section properties will be used, but the values for section size will be functions in term of x variable.

Table 1. Standard equation for section properties (Weaver, 1980, Roark, 2002)

SECTION	A	I_Z	I_Y	J	f
	$A=\pi r^2$ $=\frac{\pi}{4}D^2$	$\frac{\pi}{4}r^4$	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}$
	$A=2\pi rt$ $=\frac{\pi}{4}(Do^2-Di^2)$	$\pi r^3 t$	$\pi r^3 t$	$2\pi r^3 t$	$2.D$
	$b.h$	$\frac{bh^3}{12}$	$\frac{hb^3}{12}$	$J=\beta hb^3$ $\mathcal{R}\geq b$ $=\frac{1}{3}-0{,}21\frac{b}{h}(1-b^4/(12h^4))$	$\frac{6}{5}$
	$A=2(btf+htw)$ $=B_0.H_0-B_iH_i$	$\frac{h^2}{6}(htw+3btf)$	$\frac{h^2}{6}(htw+3btf)$	$2b^2h^2\times\frac{tftw}{btw+htf}$	$fy=\frac{A}{2htw}$ $fz=\frac{A}{2btf}$
	$A=2bt+tw.d$	$\frac{b(d+2t)^3}{12}-\frac{(b-tw)d^3}{12}$	$\frac{b^3t}{6}+\frac{tw^3d}{12}$	$\frac{1}{3}(dtw^3+2btf^3)$ $\times 1.3$	$fy=\frac{A}{2htw}$ $fz=\frac{A}{2btf}$
	$t_fb+tw.d$ $y_c=\frac{bt^2+tw.d.(2t+d)}{2(tb+twd)}$	$\frac{b}{3}(d+t)^3-\frac{d^3}{3}(b-tw)-A(d+t-y_c)^2$	$\frac{tb^3}{12}+\frac{dtw^3}{12}$	$\frac{1}{3}(b.tf^3+d.tw^3)$ $\times 1.12$	$fy=\frac{A}{htw}$ $fz=\frac{A}{2btf}$
	$2tb+tw.d$ $X_c=\frac{(2btf\frac{b}{2}+dtw.\frac{tw}{2})}{A}$	$(b(d+2tf)^3-(b-tw)d^3/12$	$\frac{(d+2tf)b^3}{3}-\frac{d}{3}(b-tw)^3-A(b+tw-x_c)^2$	$\frac{1}{3}(d'.tw^3+2b.tf^3)$	$fy=\frac{A}{d'tw}$ $fz=\frac{A}{2b.tf}$
	$t(b+d-t)$ $x_c=\frac{b^2+dt-t^2}{2(b+d-t)}$ $y_c=\frac{d^2+bt-t^2}{2(b+d-t)}$	$\frac{1}{3}[(bd^3-(b-t).(d-t)^3]-A(d-y_c)^2$	$\frac{1}{3}[(db^3-(d-t).(b-t)^3]-A(b-x_c)^2$	$\frac{1}{3}(d'.t^3+b'.t^3)$	$fy=\frac{A}{d't}$ $fz=\frac{A}{b't}$
	$bd-b_id_i$	$\frac{bd^3-b_id_i^3}{12}$	$\frac{db^3-d_ib_i^3}{12}$	$\frac{1}{3}(2.d'.tw^3+2.b'.tf^3)$	$fy=\frac{A}{2d'.tw}$ $fz=\frac{A}{2b'.tf}$

Parabolic tapered frame with h1 > h2

We will derive an equation for h in term of x for a parabolic tapered frame height with h1 > h2 as follows:

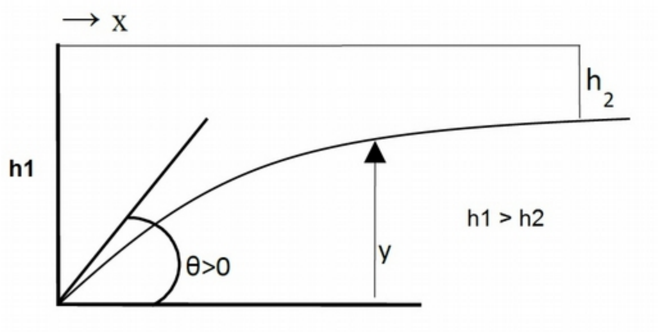


Fig.4. Parabolically Tapered Height

Assuming the boundary condition y=0 at x=0, θ > 0 for x=0, and θ=0 for x=L, we will get the following equation:

$$x = L\ ;\ \theta = 0, y = h_1 - h_2$$

$$y' = 2ax + b$$

$$x = L:\ \ y' = 2a.L + b = 0$$

$$b = -2aL$$

$$y = a.L^2 + b.L = h_1 - h_2$$

Substitute b into the equation, we will get a as follows :

$$aL^2 + (-2aL).L = h_1 - h_2$$

$$aL^2 - 2aL^2 = h_1 - h_2$$

$$-aL^2 = h_1 - h_2$$

$$a = \frac{-(h_1 - h_2)}{L^2}$$

$$b = -2aL = \frac{2(h_1 - h_2)}{L}$$

And the final equation of y in term of x is :

$$y = \frac{-(h_1 - h_2)}{L^2}x^2 + 2\frac{(h_1 - h_2)}{L}x \qquad (4)$$

using h = h1-y, we will get:

$$h = h_1 - y$$

$$h = h_1 + \frac{(h_1 - h_2)}{L^2}x^2 - 2\frac{(h_1 - h_2)}{L}x \quad (5)$$

Using the same procedure for the other case (h1 < h2), where y=h2-h1 at x=0, θ > 0 for x=L, and θ=0 for x=0, we will get the following equation for h:

$$h = h_2 + (h_2 - h_1).x^2 - (h_2 - h_1) \quad (6)$$

So, the integral equations for flexibility terms will be computed using A, J, Iy, Iz computed using h as a function of x as defined above. For certain shape, even shear area factors fy and fz will also depend on h and furthermore on x value. Different function of h and b can be derived for different cases, and in case of segmented frame, integration can be conducted for each segment with different shape modifier functions.

For h varies linearly:

h = h1 + (x/L)(h2-h1)

For frame with linear haunches at both ends with haunch length = L1 and h1 < h2 :

x < L1 : h = h2 – (x/L1) (h2-h1)
x > (L-L1) : h = h2 – ((1-x/L)/L1)(h2-h1)
(7)
L1 <= x <= (L-L1) : h = h1

For frame with parabolic haunches at both ends with haunch length = L1 and h1 < h2 :

x < L1 : h = h2 – ((h1-h2)/(L1*L1))*x*x + 2*(h1-h2)*x/L1
x > (L-L1) : h = h1+ (h2-h1)((x-(L-L1))/L1)^2
(8)
L1 <= x <= (L-L1) : h = h1

Explicit Form of Flexibility Terms

For simple tapered element, it is possible to get a not very complicated explicit formulation for flexibility terms. For example, for rectangular tapered element varies in height, we can use a symbolic math program such as MAXIMA or Mathematica, to evaluate the flexibility integrals symbolically to get the following explicit equation (Tena-Colunga, 1996) :

FM11 =

$$\left\{\frac{L\,\mathrm{ArcTan}\left[\sqrt{-1+\frac{h1}{h2}}\right]}{b\,Em\sqrt{(h1-h2)\,h2}}\right\} \quad (9)$$

FM22,b = (bending)

$$\left\{\frac{3\,L^3\left(h2\,(h1^2-3\,h1\,h2+2\,h2^2)+h1^2\sqrt{(h1-h2)\,h2}\;\mathrm{ArcTan}\left[\sqrt{-1+\frac{h1}{h2}}\right]\right)}{2\,b\,Em\,h1^2\,(h1-h2)^2\,h2^2}\right\}$$

(10)

FM22,s = (shear)

$$\left\{\frac{6\,L\,\mathrm{ArcTan}\left[\sqrt{-1+\frac{h1}{h2}}\right]}{5\,b\,G\sqrt{(h1-h2)\,h2}}\right\} \tag{11}$$

But for more complicated section, such as Wide Flange section, the resulted explicit equation, if existed, is very complex, lengthy and will require many multiplications:

FM11 =

$$\left\{-\frac{L\,\mathrm{Log}\left[\frac{2\,b\,tf+h2\,tw-2\,tf\,tw}{2\,b\,tf+h1\,tw-2\,tf\,tw}\right]}{Em\,h1\,tw-Em\,h2\,tw}\right\} \tag{12}$$

FM22,b =

$$
\begin{aligned}
\Bigg\{ & \frac{1}{3\,b^2\,Em\,(h1-h2)^3\,(h1-tf)\,tf^2\,tw} \\
& L^3 \Big(6\,b\,h1\,h2\,tf\,tw-6\,b\,h2^2\,tf\,tw-6\,b\,h1\,tf^2\,tw+6\,b\,h2\,tf^2\,tw+ \\
&\quad (h1-tf)\,(-h2+tf)\,tw\,(12\,b\,tf+(h2-tf)\,tw)\,\mathrm{Log}[h1-tf]+ \\
&\quad (h1-tf)\,(h2-tf)\,tw\,(12\,b\,tf+h2\,tw-tf\,tw)\,\mathrm{Log}[h2-tf]+ \\
&\quad 36\,b^2\,h1\,tf^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]-36\,b^2\,tf^3\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]+ \\
&\quad 12\,b\,h1\,h2\,tf\,tw\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]-12\,b\,h1\,tf^2\,tw\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]- \\
&\quad 12\,b\,h2\,tf^2\,tw\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]+12\,b\,tf^3\,tw\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]+ \\
&\quad h1\,h2^2\,tw^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]-2\,h1\,h2\,tf\,tw^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]- \\
&\quad h2^2\,tf\,tw^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]+h1\,tf^2\,tw^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]+ \\
&\quad 2\,h2\,tf^2\,tw^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]-tf^3\,tw^2\,\mathrm{Log}[6\,b\,tf+h1\,tw-tf\,tw]- \\
&\quad 36\,b^2\,h1\,tf^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]+36\,b^2\,tf^3\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]- \\
&\quad 12\,b\,h1\,h2\,tf\,tw\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]+12\,b\,h1\,tf^2\,tw\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]+ \\
&\quad 12\,b\,h2\,tf^2\,tw\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]-12\,b\,tf^3\,tw\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]- \\
&\quad h1\,h2^2\,tw^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]+2\,h1\,h2\,tf\,tw^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]+ \\
&\quad h2^2\,tf\,tw^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]-h1\,tf^2\,tw^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]- \\
&\quad 2\,h2\,tf^2\,tw^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]+tf^3\,tw^2\,\mathrm{Log}[6\,b\,tf+h2\,tw-tf\,tw]\Big)\Bigg\}
\end{aligned}
\tag{13}$$

In this case, a numerical integration will be more effective and easier to calculate. Please notice that for cases with variable form factor, more simpler equations can be approximated using constant form factor of 0.205, which is the average range of b/h ratio from 0.5 to 2.0.

Numerical Integration solution for Flexibility Terms

Among various numerical integration methods available, the constant spaced Bode's Integral and variable spaced Gauss Quadrature Integral are the most effective methods recommended. Using Bode's Integral, which is a 5 points Newton-Cotes Methods, about 64 division will be needed, while using Gauss Quadrature, 8 to 10 Gauss points will be needed. Function for Bode's rule is as follows (Pavel, 2011):

$$\frac{2h}{45}(7f_0+32f_1+12f_2+32f_3+7f_4) \tag{14}$$

Using Gauss Quadrature, we must change the integration variable from x to normalized variable s, with the following relationship:

$$x = (L/2)\,(1+s) \tag{15}$$

and

$$dx = (L/2)\,ds \tag{16}$$

The integral equation then will be executed as summation of function values evaluated at Gauss points:

$$\int_a^b w(x) f(x)\, dx = \sum_{i=1}^{n} c_i f(x_i) \tag{17}$$

Using standard Gauss Quadrature procedure with boundary from -1 to +1, we must transform the function and differential form f(x) to f(s) and from dx to ds using equation (15) and (16).

Values for Gauss points and its weight for n=1 to 10 are given below:

Table 2: High precision Gauss Constants to 25 decimal (Pavel, 2011)

High-precision Abscissae and Weights of Gaussian Quadrature.
Correctly rounded to 25 decimal digits to the nearest.

n	Abscissae ξ_i	Weights w_i
2	±0.5773502691896257645091488	1.0000000000000000000000000
3	0	0.8888888888888888888888889
	±0.7745966692414833770358531	0.5555555555555555555555556
4	±0.3399810435848562648026658	0.6521451548625461426269361
	±0.8611363115940525752239465	0.3478548451374538573730639
5	0	0.5688888888888888888888889
	±0.5384693101056830910363144	0.4786286704993664680412915
	±0.9061798459386639927976269	0.2369268850561890875142640
6	±0.2386191860831969086305017	0.4679139345726910473898703
	±0.6612093864662645136613996	0.3607615730481386075698335
	±0.9324695142031520278123016	0.1713244923791703450402961
7	0	0.4179591836734693877551020
	±0.4058451513773971669066064	0.3818300505051189449503698
	±0.7415311855993944398638648	0.2797053914892766679014678
	±0.9491079123427585245261897	0.1294849661688696932706114
8	±0.1834346424956498049394761	0.3626837833783619829651504
	±0.5255324099163289858177390	0.3137066458778872873379622
	±0.7966664774136267395915539	0.2223810344533744705443560
	±0.9602898564975362316835609	0.1012285362903762591525314
9	0	0.3302393550012597631645251
	±0.3242534234038089290385380	0.3123470770400028400686304
	±0.6133714327005903973087020	0.2606106964029354623187429
	±0.8360311073266357942994298	0.1806481606948574040584720
	±0.9681602395076260898355762	0.0812743883615744119718922
10	±0.1488743389816312108848260	0.2955242247147528701738930
	±0.4333953941292471907992659	0.2692667193099963550912269
	±0.6794095682990244062343274	0.2190863625159820439955349
	±0.8650633666889845107320967	0.1494513491505805931457763
	±0.9739065285171717200779640	0.0666713443086881375935688

Derivation of Stiffness Matrix from Flexibility Matrix

The 12x12 Stiffness Matrix of an element can be derived from The 6x6 Flexibility matrix using a simple procedure as follows (Weaver, 1980):

First, we will derive Stiffness sumatrix Smkk as matrix inverse of Fmkk :

$$Smkk = [Fmkk]^{-1} \qquad (18)$$

The inverse of Fmkk is given below:

$$Smkk = \begin{pmatrix} \frac{1}{FM11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{FM66}{-FM26^2+FM22\ FM66} & 0 & 0 & 0 & \frac{FM26}{FM26^2-FM22\ FM66} \\ 0 & 0 & \frac{FM55}{-FM35^2+FM33\ FM55} & 0 & \frac{FM35}{-FM35^2+FM33\ FM55} & 0 \\ 0 & 0 & 0 & \frac{1}{FM44} & 0 & 0 \\ 0 & 0 & \frac{FM35}{-FM35^2+FM33\ FM55} & 0 & \frac{FM33}{-FM35^2+FM33\ FM55} & 0 \\ 0 & \frac{FM26}{FM26^2-FM22\ FM66} & 0 & 0 & 0 & \frac{FM22}{-FM26^2+FM22\ FM66} \end{pmatrix} \qquad (19)$$

The complete Stiffness Matrix of a space frame element is :

$$SM = \begin{pmatrix} SMjj & SMjk \\ SMkj & SMkk \end{pmatrix} \qquad (19)$$

The other three submatrices can be found by the type of axis transformations and static equilibrium, using tranformation matrix Tjk as follows (Weaver, 1980) :

$$Tjk = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -L & 0 & 1 & 0 \\ 0 & L & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (20)$$

$$Smjk = -Tjk\ Smkk \qquad (21)$$

$$\begin{pmatrix} -\frac{1}{FM11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{FM66}{FM26^2-FM22\ FM66} & 0 & 0 & 0 & \frac{FM26}{-FM26^2+FM22\ FM66} \\ 0 & 0 & \frac{FM55}{FM35^2-FM33\ FM55} & 0 & \frac{FM35}{FM35^2-FM33\ FM55} & 0 \\ 0 & 0 & 0 & -\frac{1}{FM44} & 0 & 0 \\ 0 & 0 & \frac{FM35-FM55\ L}{FM35^2-FM33\ FM55} & 0 & \frac{FM33-FM35\ L}{FM35^2-FM33\ FM55} & 0 \\ 0 & \frac{FM26-FM66\ L}{-FM26^2+FM22\ FM66} & 0 & 0 & 0 & \frac{FM22-FM26\ L}{FM26^2-FM22\ FM66} \end{pmatrix}$$

$$Smkj = Smjk^T = -Smkk\ Tjk^T \qquad (22)$$

and

$$Smjj = -Tjk\ Smkj = Tjk\ Smkk\ Tjk^T \tag{23}$$

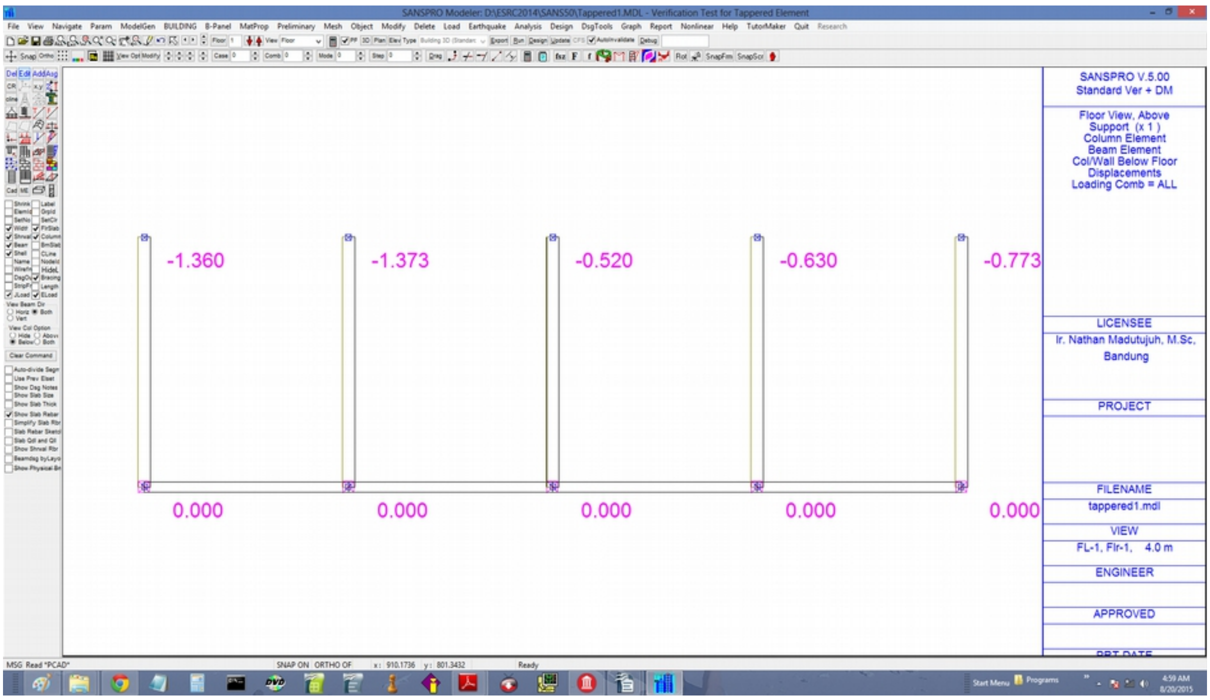
In Tena-Colunga's paper (1996), different equations for Smjk and Smjj are given, which can give incorrect stiffness matrix compared to the matrix transformation method used above. Smjk is not necessary to be symmetric, but the final stiffness matrix SM is always symmetric.

Smjj : (24)

$$\begin{pmatrix} \frac{1}{FM11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{FM66}{-FM26^2+FM22\ FM66} & 0 & 0 & 0 & \frac{FM26-FM66\ L}{FM26^2-FM22\ FM66} \\ 0 & 0 & \frac{FM55}{-FM35^2+FM33\ FM55} & 0 & \frac{FM35-FM55\ L}{-FM35^2+FM33\ FM55} & 0 \\ 0 & 0 & 0 & \frac{1}{FM44} & 0 & 0 \\ 0 & 0 & \frac{FM35-FM55\ L}{-FM35^2+FM33\ FM55} & 0 & \frac{FM33+L\ (-2\ FM35+FM55\ L)}{-FM35^2+FM33\ FM55} & 0 \\ 0 & \frac{FM26-FM66\ L}{FM26^2-FM22\ FM66} & 0 & 0 & 0 & \frac{FM22+L\ (-2\ FM26+FM66\ L)}{-FM26^2+FM22\ FM66} \end{pmatrix}$$

Case Study

A 6m concrete cantilever beam B30/60 with various tapered conditions loaded with distributed load of 1000 kg/m are analyzed using the above formulation and the tip displacement results are shown in the graph and Table 3 below.



The standard Hermitian formulation without shear deformation gives tip deflection of -1.360 cm, while if we use shear deformation, the tip deflection will be -1.373 cm. The third beam is linearly tapered height beam with tip displacement of -0.520cm, and the fourth one is parabolically tapered element with -0.630 cm displacement. The fifth beam is using linearly tapered width that gives -0.773 cm deflection.

All tapered beams have been analyzed using tapered element derived from general formulation given above. Using this approach, we can evaluate various type of tapered shape by changing only certain parameters controlling the tapered shape

Table 3. Tip displacements of various cantilever beams

No.	Description	b1 (cm)	h1 (cm)	b2 (cm)	h2 (cm)	Displacement (cm)
1	Standard Beam, no shear deformation	30	60	30	60	-1.36
2	Standard Beam, with shear deformation	30	60	30	60	-1.37
3	Linearly Tapered Height Beam	30	90	30	60	-0.52
4	Parabolically Tapered Height Beam	30	90	30	60	-0.63
5	Linearly Tapered Width Beam	60	60	30	60	-0.77

Discussion

The given procedure above is straightforward and easily to apply to various type of section and tapered pattern. The procedure also provide two alternative methods to compute the required stiffness matrix, the explicit form using symbolic math program, and numerical method using Gauss Integration. The use of 8 points Gauss Integration is recommended to get accurate solution to 16 digits.

From above study case, it is found that a tapered element is very effective in reducing tip displacement with only small increase of weight. The above formulation also allow engineers to use only 1 element to model accurately a complex shape tapered element. This approach is very useful in large steel building with all beam members having “voute” (haunch beam or tapered beam) at both ends, beams with honeycomb web shape, or beams with reduced flange shape. Without using this approach, each beam will require at least twice number of nodes, that will increase analysis time and storage space.

Conclusions

1. For a complex tapered shape element, we can use a single element stiffness modeled derived from flexibility matrix using general formulation given above

2. The single element stiffness model will give accurate result compared to segmented model approach without adding extra DOF

3. Explicit form of stiffness matrix for certain shape can be derived from general formulation above using Symbolic Math Program

4. More complex section type and tapered shape may not have explicit form and/or more effective to be solved using Gauss Quadrature Numerical Integration

5. At least 8-points Gauss Quadrature Numerical Integration must be used to get result with 16 digits accuracy

Recommendations and Further Study

1. Stiffness matrix for single element frame with tapered shape is now available and can be used without additional DOF

2. Use the given stiffness matrix for analyzing beams at both ends, or reduce flange width, or beams with honeycomb width, to get more accurate results efficiently

3. Extend the general formulation above to compute frame element end forces to get the equilibrium of forces

4. Extend the general formulation above for space frame element with linear/nonlinear nodal moment spring

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1989-Now	Development of computer program SANSPRO, PCAD, SOILAB
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